Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 6

Due Date: 26.05.2014

Exercise 1:
Let $f : X \to Y$ be a morphism of finite presentation and let $x \in X$ be a point with image $y = f(x) \in Y$. Show that f is unramified at x if and only if $\kappa(x)$ is a separable extension of $\kappa(y)$ and $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_{X,x} \subset \mathcal{O}_{X,x}$. Here $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ (resp. $\mathfrak{m}_y \subset \mathcal{O}_{Y,y}$) is the maximal ideal.
(*Hint: For the difficult direction show that the diagonal is an open immersion.*)

Exercise 2:

(i) Let k be a field and let $n \geq 1$ be prime to the characteristic of k. Let $\mathbb{G}_m = \operatorname{Spec} k[T, T^{-1}]$. Show that that the morphism $\mathbb{G}_m \to \mathbb{G}_m$ defined by $T \mapsto T^n$ is étale.

(ii) Let k be a field of characteristic p and let $\mathbb{A}^1_k = \operatorname{Spec} k[T]$. Show that the morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ defined by $T \mapsto T^p - T$ is étale.

(iii) Let A be a ring and let $f \in A[T]$. Let $B = A[T]/(f)$. Show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale if $f' = \frac{df}{dT} \in A[T]$ becomes a unit in B.
(*Hint: First compute* $\Omega^1_{B/A}$.)

Exercise 3:
Let k be an algebraically closed field and let $f : X \to Y$ be a morphism of smooth k-schemes. Show that the following are equivalent:

(a) f is smooth.

(b) $\Omega^1_{X/Y}$ is locally free.

(c) for all $x \in X(k)$ and $y = f(x) \in Y(k)$ the induced map on tangent spaces $T_x X \to T_y Y$ is surjective.

Exercise 4:
Let k be a perfect field and X be a curve over k, i.e. X is an integral k-scheme of finite type which one-dimensional. Show that X is smooth at a closed point $x \in X$ if and only if the local ring $\mathcal{O}_{X,x}$ is a principal ideal domain.
(*Hint: for the difficult direction let* $f \in \mathcal{O}_{X,x}$ *be a generator of the maximal ideal that is defined in a neighborhood* U *of* x. *Show that the morphism* $U \to \mathbb{A}^1_k$ *that is defined by* f *is étale at* x.)

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`