ARITHMETISCHE GEOMETRIE OBERSEMINAR

BONN, SOMMERSEMESTER 2011

PROGRAMMVORSCHLAG: P. SCHOLZE

TOPIC: PERFECTOID SPACES

The aim of the seminar is to understand the notion of perfectoid spaces, and its applications to the weight-monodromy conjecture, and to p-adic Hodge theory. These are, respectively, the content of [6], [8], and [7]. Other important references are the book of Gabber-Ramero on Almost ring theory, [3], the book of Huber on adic spaces, [4], and the book of Brinon on relative p-adic Hodge theory, [2]. Further, the work of Kedlaya-Liu, [5], is closely related to the statements proved in [6], and the arguments of [7] are inspired by the work of Andreatta-Iovita, [1], in the crystalline case.

0. Talk: Introduction

Explanation of main results.

1. Talk: Almost ring theory

Explain some portions of the book by Gabber-Ramero, [3]. Give the definition of an almost K^+-module and an almost K^+-algebra, as in Section 2.2, in the case where K is a non-discrete valued field with valuation of rank 1, with valuation ring K^+ and maximal ideal $\mathfrak{m}$: Some statements simplify in this situation, e.g. $\mathfrak{m} = \tilde{\mathfrak{m}}$, $\mathfrak{m}$ is flat, and condition (A) of Subsection 2.1.6 is satisfied. Moreover, any finitely generated subideal of $\mathfrak{m}$ is principal. Also explain the functor $M \mapsto M_*$ on almost modules and $A \mapsto A_*$ on almost algebras.

Further, give the definition of an almost finitely presented module, Definition 2.3.8 (v), or better use the equivalent formulation given in Corollary 2.3.13. Also give the definition of flat and almost projective modules, Definition 2.4.4. Define unramified morphisms using Proposition 3.1.4, and hence étale morphisms as unramified and flat. Finally, discuss the proof of Theorem 3.5.28.

2. Talk: Perfectoid rings

Give the definition of a perfectoid ring as in [6] and cover Section 1: In particular, prove the tilting equivalence between perfectoid rings in characteristic 0 and in characteristic p and show that it induces a fully faithful functor on finite étale covers, and in fact an equivalence if the perfectoid ring is a field.

3. Talk: Perfectoid Spaces: Analytic topology

First, give a quick review of the definition of an analytic adic space locally of finite type over K, as in Huber's book [4], Section 1.1. This allows for several simplifications: All our rings are complete Tate rings, and in fact quotients of $K\langle T_1, \ldots, T_n\rangle$, and satisfy condition (1.1.1) b). The rings of integral elements are always given by the subring of power-bounded elements. The topology can be defined using rational subsets, and the condition $|g(x)| \neq 0$ can be dropped. Further, all points are analytic, and hence all specializations and generalizations secondary. It would be nice to see the picture of $\mathbb{P}^1_K$ in case K is algebraically closed. Then give the definition of a perfectoid space and cover Section 2 of [6]: In particular, study their structure sheaves, prove that they are identified under tilting, and that tilting induces a homeomorphism of underlying topological spaces.

4. Talk: Perfectoid Spaces: Etale topology

Cover the rest, i.e. Section 3, of [6]. First, recall the definition of an étale map of adic spaces, Definition 1.6.5 i), and state Example 1.6.6 ii) and Lemma 2.2.8 of [4]. These motivate the definition of étale maps of perfectoid spaces. Then give the definition of an étale map of perfectoid spaces, prove the almost purity theorem, and the equivalence of etale topoi under tilting. Finally, relate the etale topoi of perfectoid spaces to etale topoi of adic spaces.

5. Talk: The weight-monodromy conjecture

Recall the statement of the weight-monodromy conjecture and prove it in the case of a smooth hypersurface, cf. [8].

6. Talk: p-adic Hodge theory: The pro-étale site and sheaves of relative periods

Give the definition of the pro-étale site of a quasicompact quasiseparated smooth adic space X over K, introduce the sheaves of relative periods, and give an explicit description of them as in Sections 1 – 3 of [7], but omit the statement $\nu_* \mathbb{B}_{\mathrm{dR}} = \mathcal{O}_X$. Explain the relationship to Fontaine's rings in the case $X = \mathrm{Spa}(K)$.

7. Talk: p-adic Hodge theory: The deep comparison isomorphism

Explain Section 4 of [7], and prove $\nu_* \mathbb{B}_{\mathrm{dR}} = \mathcal{O}_X$, which is necessary for Proposition 4.8. For this, one will have to recall some arguments from [2].

8. Talk: p-adic Hodge theory: Comparison of deep and usual deRham cohomology

Prove the comparison of deep and usual deRham cohomology, as in Section 6 of [7]. Also explain the GAGA theorems, cf. Section 5.

9. Talk: p-adic Hodge theory: Conclusion

Deduce the deRham comparison isomorphism in p-adic Hodge theory, with coefficients and in families, as in Section 7 of [7]. Explain the 'tricks' used to show that any reasonable map between reasonable cohomology theories is an isomorphism.

References

[1] F. Andreatta and A. Iovita. Comparison Isomorphisms for Formal Schemes. http://www.mathstat.concordia.ca/faculty/iovita/paper_14.pdf.
[2] O. Brinon. Représentations p-adiques cristallines et de de Rham dans le cas relatif. *Mém. Soc. Math. Fr. (N.S.)*, (112):vi+159, 2008.
[3] O. Gabber and L. Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.
[4] R. Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.
[5] K. Kedlaya and R. Liu. Relative p-adic Hodge theory, I: Foundations. 2011. http://math.mit.edu/~kedlaya/papers/relative-padic-Hodge1.pdf.
[6] P. Scholze. Perfectoid Spaces, I: Basic theory. 2011. in preparation.
[7] P. Scholze. Perfectoid Spaces, II: p-adic Hodge theory. 2011. in preparation.
[8] P. Scholze. Perfectoid Spaces, III: On the weight-monodromy conjecture. 2011. in preparation.