# Tensor products of irreducible $\mathfrak{sl}_2$-modules: Decomposition rules and endomorphism rings

MAX-PLANCK-GESELLSCHAFT

universitätbonn


Meinel, Joanna

Advisor : Prof. Dr. Catharina Stroppel


## Background knowledge

### Reminder on $\mathfrak{sl}_2$-modules

Recall the Lie algebra $\mathfrak{sl}_2$ of traceless $2 \times 2$-matrices over the complex numbers: It is spanned by $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the Lie bracket is given by the matrix commutator $[X,Y] = XY - YX$. The generators satisfy the relations $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$. We want to study finite dimensional modules over $\mathfrak{sl}_2$. The following facts make our life easy:

- Every finite dimensional $\mathfrak{sl}_2$-module decomposes into a direct sum of irreducible modules (Weyl's theorem on complete reducibility)
- We know exactly how the finite dimensional irreducible $\mathfrak{sl}_2$-modules look like: They are (up to isomorphism) uniquely determined by their dimension, and the $n+1$-dimensional one is given by

$$V_n = \operatorname{span}_{\mathbb{C}} \{v_n, v_{n-2}, \ldots, v_{-n+2}, v_{-n}\},$$

where $v_k$ is of weight $k$, ie. $H(v_k) = k \cdot v_k$. The module $V_n$ is highest weight of weight $n$.

For example, the 1-dimensional $\mathfrak{sl}_2$-module $V_0 = \operatorname{span}_{\mathbb{C}}\{v_0\}$ is the *trivial* module: As vector space, it is isomorphic to $\mathbb{C}$, and $\mathfrak{sl}_2$ acts by 0. The 2-dimensional $\mathfrak{sl}_2$-module $V_1 = \operatorname{span}_{\mathbb{C}}\{v_1, v_{-1}\}$ is called the *natural* module: It is $\mathbb{C}^2$ with action by matrix-vector-multiplication. The tensor product $V \otimes W$ of two $\mathfrak{sl}_2$-modules is again a $\mathfrak{sl}_2$-module with action $X(v \otimes w) = X(v) \otimes w + v \otimes X(w)$.

Aim: Decompose tensor products $V_{d_1} \otimes \ldots \otimes V_{d_r}$ into irreducible modules!

Describe the endomorphism ring $\operatorname{End}_{\mathfrak{sl}_2}(V_{d_1} \otimes \ldots \otimes V_{d_r})$!

### Clebsch-Gordon decomposition rule for tensor products of $\mathfrak{sl}_2$-modules

For the irreducible modules $V_n$ and $V_m$, say $n \geq m$, we have

$$V_n \otimes V_m \;\cong\; V_{n+m} \oplus V_{n+m-2} \oplus \ldots \oplus V_{n-m+2} \oplus V_{n-m}.$$

So we get for example $V_1 \otimes V_1 \;\cong\; V_2 \oplus V_0$ and hence $V_1 \otimes V_1 \otimes V_1 \;\cong\; (V_2 \oplus V_0) \otimes V_1 \;\cong\; V_3 \oplus V_1 \oplus V_1$. Successive application of this rule enables us to decompose every $V_{d_1} \otimes \ldots \otimes V_{d_r}$ by hand, see also [Kas95]. Notice furthermore that every irreducible module $V_d$ appears in the tensor product $V_1^{\otimes d} = \underbrace{V_1 \otimes \ldots \otimes V_1}_{d \text{ times}}$.

### Classical multiplicity formulas

What is the *multiplicity* of $V_k$ in $V_1^{\otimes d}$? We want to determine the numbers $m_k^d$ in $V_1^{\otimes d} = \bigoplus V_k^{\oplus m_k^d}$. There is eg. a formula by Lehrer-Zhang [LZ10]:

$$m_k^d = \binom{d}{\frac{d+k}{2}} \frac{2(k+1)}{d+k+2} \quad \text{for } d \geq 1 \text{ and } 0 \leq k \leq d.$$

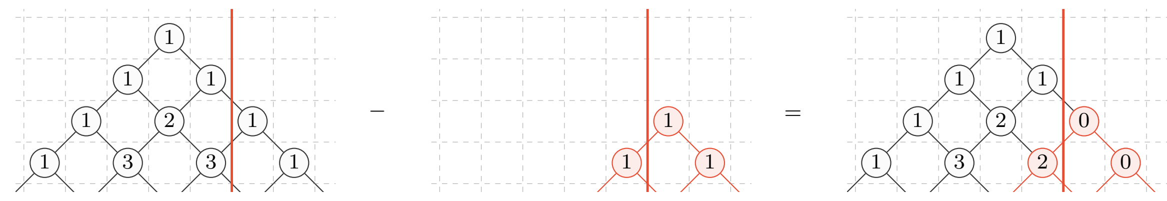
From the Clebsch-Gordon rule we can read off: This multiplicity is also given by the number of possibilities to write $k$ as a sum $\sum_1^d \pm 1$ of $d$ summands, each either $+1$ or $-1$, such that each partial sum is nonnegative: $\sum_1^m \pm 1 \geq 0$ for all $1 \leq m \leq d$.

## Another multiplicity formula using Catalan numbers

Successive application of the Clebsch-Gordon rule gives the following multiplicities $m_k^d$ of $V_k$ in $V_1^{\otimes d}$:

| | $V_{10}$ | $V_9$ | $V_8$ | $V_7$ | $V_6$ | $V_5$ | $V_4$ | $V_3$ | $V_2$ | $V_1$ | $V_0$ | $V_{-1}$ | $V_{-2}$ | $V_{-3}$ | $V_{-4}$ | $V_{-5}$ | $V_{-6}$ | $V_{-7}$ | $V_{-8}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $V_1^{\otimes 1}$ | | | | | | | | | | 1 | | | | | | | | | |
| $V_1^{\otimes 2}$ | | | | | | | | | 1 | | 1 | | | | | | | | |
| $V_1^{\otimes 3}$ | | | | | | | | 1 | | 2 | | 0 | | | | | | | |
| $V_1^{\otimes 4}$ | | | | | | | 1 | | 3 | | 2 | | 0 | | | | | | |
| $V_1^{\otimes 5}$ | | | | | | 1 | | 4 | | 5 | | 0 | | 0 | | | | | |
| $V_1^{\otimes 6}$ | | | | | 1 | | 5 | | 9 | | 5 | | 0 | | 0 | | | | |
| $V_1^{\otimes 7}$ | | | | 1 | | 6 | | 14 | | 14 | | 0 | | 0 | | 0 | | | |
| $V_1^{\otimes 8}$ | | | 1 | | 7 | | 20 | | 28 | | 14 | | 0 | | 0 | | 0 | | |

This triangle is very similar to a Pascal triangle: Almost all its entries are the sum of the two entries from above, except for the fact that there are no summands belonging to a negative weight in $V_1^{\otimes d}$, so one has to delete every entry on the right hand side of the red line (which would correspond to a negative weight). This error is propagated throughout the triangle. But one can describe the propagation of the error via further Pascal triangles that have to be subtracted from the original one. Take a look at the top of the triangle:



We see that we have to subtract a whole (translated) Pascal triangle to compensate for the error coming from the 'wrong 1'. It is the multiplicity of $V_0$ in $V_1^{\otimes 2}$. We have to repeat this in every second row: For $d$ even, subtract a Pascal triangle with multiplicity = the multiplicity of the trivial representation $V_0$ in $V_1^{\otimes d}$.

### Catalan numbers

This multiplicity equals the $\frac{d}{2}$-th Catalan number. The Catalan numbers $C_n = \binom{2n}{n}\frac{1}{n+1}$ appear in numerous combinatorial problems! They count eg. triangulations of a convex $(n+2)$-gon, or paths in the plane from $(0,0)$ to $(2n,0)$ using only steps $(1,1)$ and $(1,-1)$ never passing below the $x$-axis, see [Sta99]. We use the interpretation of $C_n$ as number of sequences of $n$ 1's and $n$ $(-1)$'s with nonnegative partial sums.

### The formula

The multiplicity of $V_{d-2k}$ in $V_1^{\otimes d}$ is given by $m_{d-2k}^d = \binom{d-1}{k} - \sum_{j=1}^{d-1} C_j \binom{d-1-2j}{k-(j+1)}$, where we use the convention $\binom{n}{k} = 0$ whenever $n < 0$, $k < 0$ or $k > n$ (some summands vanish, the sum runs up to $d-1$ only for aesthetical reasons). With this convention the formula holds for all $d$ and $k$, even silly ones.

## Decomposition of $V_{d_1} \otimes \ldots \otimes V_{d_r}$

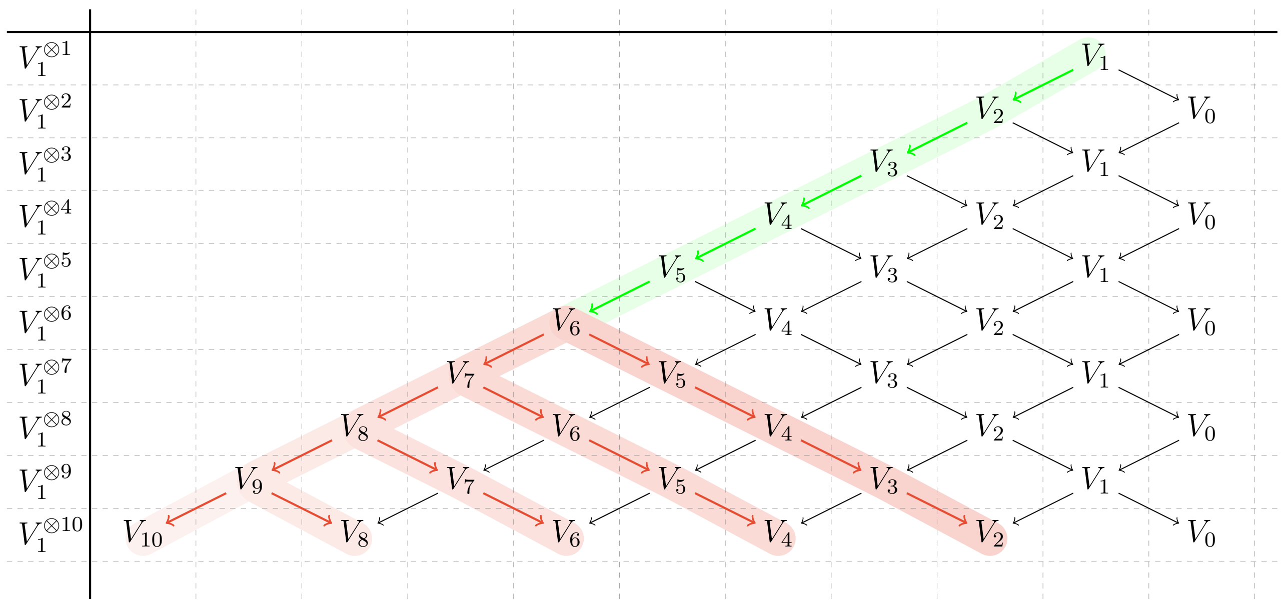
We want to describe

$$V_{d_1} \otimes \ldots \otimes V_{d_r} \;\subset\; V_1^{\otimes d_1} \otimes \ldots \otimes V_1^{\otimes d_r} \;=\; V_1^{\otimes d_1 + \ldots + d_r}.$$

Problem: This inclusion is not canonical, since the decomposition of $V_1^{\otimes d}$ into irreducible summands is not canonical (only the isotypical components are). But at least

$$V_k \otimes V_1 \;=\; V_{k+1} \oplus V_{k-1} \qquad \text{for } k \geq 1$$

is unique. So if we view $V_1^{\otimes d}$ as the result of successive tensor products $V_1 \otimes V_1$, $V_1^{\otimes 2} \otimes V_1$, …, $V_1^{\otimes d-1} \otimes V_1$, we have in each step control of the decomposition. This enables us to trace down certain irreducible summands following a path in the decomposition picture:




Each arrow is a (canonical) projection onto $V_{k+1}$ or $V_{k-1}$ inside $V_k \otimes V_1$. Projecting along a path, ie. composing the projections read off from the path, gives a certain summand inside $V_1^{\otimes d}$.

### Highest weight vectors

So it makes sense to ask: How do $V_{k+1}$ and $V_{k-1}$ lie exactly inside $V_k \otimes V_1$? We answer this in terms of highest weight vectors: Denote by $v = v_k$ the highest weight vector of $V_k$. For the irreducible $\mathfrak{sl}_2$-representations $V_k$ and $V_1$, the two linearly independent highest weight vectors in $V_k \otimes V_1$ are (up to scalars) obtained by

$\swarrow : v \;\mapsto\; v \otimes v_1$ which is of weight $k+1$,

$\searrow : v \;\mapsto\; EF(v) \otimes v_{-1} - F(v) \otimes v_1$ which is of weight $k-1$.

In other words, we have an explicit decomposition of $V_k \otimes V_1$ into

$$V_k \otimes V_1 \;=\; \mathfrak{U}(\mathfrak{sl}_2)(v \otimes v_1) \;\oplus\; \mathfrak{U}(\mathfrak{sl}_2)(EF(v) \otimes v_{-1} - F(v) \otimes v_1).$$

Now we are ready to agree on a convention to choose a path corresponding to $V_{d_1} \otimes \ldots \otimes V_{d_r}$ inside $V_1^{\otimes d_1 + \ldots + d_r}$. For example, always take the leftmost path, like in the above example for $V_6 \otimes V_4$.

## Endomorphism rings

The endomorphism ring $\operatorname{End}_{\mathfrak{sl}_2}(V_1^{\otimes d})$ is characterized by classical Schur-Weyl duality as summand of $\mathbb{C}[\mathfrak{S}_d]$ (the group algebra of the symmetric group) because there are no $\mathfrak{sl}_2$-morphisms other than the permutation of the tensor factors. What about $\operatorname{End}_{\mathfrak{sl}_2}(V_{d_1} \otimes \ldots \otimes V_{d_r})$? Thanks to Schur's lemma, these endomorphisms are nothing but block matrices. Nevertheless, we aim at a description of

$$\operatorname{End}_{\mathfrak{sl}_2}(V_{d_1} \otimes \ldots \otimes V_{d_r}) \;\subset\; \operatorname{End}_{\mathfrak{sl}_2}(V_1^{\otimes d_1 + \ldots + d_r}),$$

using the description of $V_{d_1} \otimes \ldots \otimes V_{d_r}$ via paths. The paths represent projections to the summands, as we have seen. These considerations are a special case of higher Schur-Weyl duality as discussed in [BK08].

### The Hecke algebra

Higher Schur-Weyl duality relates some $\operatorname{End}_{\mathfrak{gl}_n}(M^{\mathfrak{p}} \otimes V^{\otimes d})$ with the degenerate affine Hecke algebra. This Hecke algebra is generated by $\mathbb{C}[x_1, \ldots, x_d]$ and $\mathbb{C}[\mathfrak{S}_d]$ with additional relation $s_i x_{i+1} = x_i s_i + 1$. There is another graphical description in terms of KLR-generators (coloured strands and fat dots!) which provides idempotents projecting to the summands of $M^{\mathfrak{p}} \otimes V^{\otimes d}$. Back in our setting, this means that we have to add all the idempotents that project to the summands of $V_{d_1} \otimes \ldots \otimes V_{d_r}$ to get an idempotent $e$ such that

$$\operatorname{End}_{\mathfrak{sl}_2}(V_{d_1} \otimes \ldots \otimes V_{d_r}) \;=\; e \cdot \operatorname{End}_{\mathfrak{sl}_2}(V_1^{\otimes d_1 + \ldots + d_r}) \cdot e \;\subset\; \operatorname{End}_{\mathfrak{sl}_2}(V_1^{\otimes d_1 + \ldots + d_r}).$$

### Outlook

Recent research deals with all kinds of endomorphism rings of tensor products. One looks for generalizations for other types as in [LZ]. Or one asks the same questions for the quantum group $\mathfrak{U}_q(\mathfrak{sl}_2)$: At least one can decompose the tensor product $V_k \otimes V_1$ (the irreducible type $+1$ representations of dimension $k+1$ resp. 2) in the same way as we did for $\mathfrak{sl}_2$, one only needs to add one extra $q$ to compensate for the asymmetric action of $\mathfrak{U}_q(\mathfrak{sl}_2)$ on the tensor product: The two highest weight vectors in $V_k \otimes V_1$ are $v \otimes v_1$ of weight $q^{k+1}$ and $EF(v) \otimes v_{-1} - qF(v) \otimes v_1$ of weight $q^{k-1}$. See also the pictures in [BH11]! But eg. the description of $\operatorname{End}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(V_{d_1} \otimes \ldots \otimes V_{d_r})$ is still pending.

### References


[BH11] G. Benkart and T. Halverson, *Motzkin algebras*, Preprint.

[BK08] J. Brundan and A. Kleshchev, *Schur-Weyl duality for higher levels*, Selecta Math. **14** (2008), 1–57.

[Kas95] C. Kassel, *Quantum groups*, Springer, 1995.

[LZ] G. Lehrer and R. Zhang, *The second fundamental theorem of invariant theory for the orthogonal group*, to appear in Annals of Math.

[LZ10] ———, *A Temperley-Lieb analogue for the BMW algebra*, Progr. Math., vol. 284, 2010, pp. 155–190.

[Sta99] R. P. Stanley, *Enumerative combinatorics 2*, Cambridge University Press, 1999.



2012 Max-Planck-Institut für Mathematik
joanna.meinel@uni-bonn.de