Def: A log pair (X, D) has (Kawamata) log terminal singularities iff

i) $K_X + D$ is $\mathbb{Q}$-Cartier

ii) $\exists\, f : Y \to X$ birational, proper, Y smooth s.t. D_Y has nc and in the ramification formula $K_Y + D_Y = f^*(K_X + D) + \sum a_i E_i$ one has $a_i > 0$

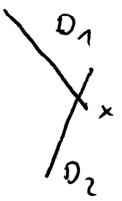
iii) $0 \leq d_i < 1$ if $D = \sum d_i D_i$, i.e.

$$[D] = 0.$$

Rem: • Again we may instead write $K_Y = f^*(K_X + D) + \sum b_i F_i$ and require $b_i > -1$ for all exceptional F_i

• Note that ~~thus~~ without the condition iii) one could in fact not assure that ii) then holds for any resolution $f' : Y' \to X$.

Here is an explicit example: Let X be a smooth surface and $D = D_1 + D_2 \subset X$ a snc with $D_1 \cap D_2 = \{x\}$



$(d_1 = d_2 = 1)$. Consider $f = id : Y = X \to X$ as a desing. Then (X, D) does satisfy ii).

If on the other hand $f' : Y' \to X$ is the blow-up of $x \in X$, then $K_{Y'} = f'^* K_X + E = f'^*(K_X + D) - \hat{D} - E$, i.e. $b = -1$.

(Use $f^* D_1 = \hat{D}_1 + E$.)

• If X is smooth and $D = \sum d_i D_i$ with $0 \le d_i < 1$ is a nc divisor, then (X, D) is log terminal

<u>Example</u> : See [Matsuki] page 248

Suppose $\dim(X) = 2$, $D = \sum D_i$ $(d_i = 1)$
is log terminal at $x \in X$

• either X smooth in x
and $D = D_1$ or $D = D_1 + D_2$ nc in x

• or X has in $x \in X$ a quotient singularity of type A_n
$D = D_1$ and $x \in D$ is a smooth point of D

<u>Discrepancies</u> : Let $(X, D = \sum d_i D_i)$ be given such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $f : Y \to X$ is birational proper $\leadsto$ $K_Y = f^*(K_X + D) + \sum b_i F_i$

If $F = F_i$ is exceptional, one defines

$\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\text{discrep}(X, D, F)} := b_i$

Then $\underset{\sim\sim\sim\sim}{\text{discrep}}(X, D) = \inf \{ \text{discrep}(X, D, F) \mid$
$F = $ except. divisor of some
$f : Y \to X$ proper, birational $\}$

Then [Kollár, Mori] show

- either $\operatorname{discrep}(X, D) = -\infty$ (e.g. if some $d_i > 1$)
- or $\operatorname{discrep}(X, D) \in [-1, 1]$

Moreover, $(X, D = \sum d_i D_i$ effective$)$ is

log canonical $\iff$ $\operatorname{discrep}(X, D) \geq -1$

(automatically $d_i \leq 1$)

Kawamata log terminal $\iff$ $\operatorname{discrep}(X, D) > -1$

and $\lfloor D \rfloor = 0$

Notation: $d \in \mathbb{Q}$ $\quad \lfloor d \rfloor = \max\{n \in \mathbb{Z} \mid n \leq d\}$

$\cancel{\lfloor d \rfloor} = \lceil d \rceil = \min\{n \in \mathbb{Z} \mid d \leq n\}$

$\{d\} = d - \lfloor d \rfloor$

$\rightsquigarrow D = \sum d_i D_i \quad \rightsquigarrow \quad \lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$

$\lceil D \rceil = \sum \lceil d_i \rceil D_i$

$\{D\} = \sum \{d_i\} D_i$

How to resolve

We shall list various versions of how to resolve singularities without giving any details. Originally this goes back to Hironaka, but recently easier proofs or different approaches have been invented by Bierstone/Milman, Villamayor (+ Bravo, Encinas), de Jong (+ Abramovich, Bogomolov/Pantev, Paranjape), Włodarczyk (+ Abramovich, Kalle, Matsuki).

X = (quasi-) projective variety over $k = \bar{k}$ (char $k = 0$) or even $k = \mathbb{C}$, normal

(Often it works for analytic varieties.)

1 a) A <u>resolution</u> of X consists of

$f : Y \to X$ birational, proper (projective)

s.t. Y is smooth.

b) A <u>strong resolution</u> is a resolution $f : Y \to X$ with
- $f : Y \setminus f^{-1}(X_{sing}) \cong X \setminus X_{sing}$
- $f^{-1}(X_{sing})$ nc divisor
- f is a sequence of blow-ups with smooth centers in X_{sing}

(In general, "alterations" of de Jong do not suffice to prove the strong versions.)

2. Suppose X is smooth and $D = \sum D_i$ a divisor.

a) A <u>resolution of D</u> is a proper birational morphism $f: Y \to X$ with Y smooth and $\hat{D}$ a snc divisor. ($\hat{D}$ = strict transform)

b) A <u>strong resolution of D</u> is a resolution of D $f: Y \to X$ s.t.

- $f: f^{-1}(U) \cong U$, with $U \subset X$ the largest open set s.t. $D \cap U$ is snc
- $\hat{D} \cup \mathrm{Exc}(f)$ is a snc divisor
- f is a sequence of blow-ups with smooth centers.

<u>Rem</u>: • This can be used to prove that any smooth variety X can be completed $X \subset \bar{X}$ s.t. $\bar{X} \setminus X$ is a snc divisor. In order to do this one needs "<u>Principalization</u>": X smooth, $\mathcal{I} \subset \mathcal{O}_X$ ideal sheaf. Then there exists $f: Y \to X$ projective, birational, Y smooth s.t. $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y \subset \mathcal{O}_Y$ is invertible.

A <u>strong principalization</u> requires f to be a sequence of smooth blow-ups.

• A consequence of principalization is "<u>elimination of singularities</u>":

$g: X \dashrightarrow \mathbb{P}^N$ rational map

Then
$$\begin{tikzcd} X \arrow[r, dashed, "g"] & \mathbb{P}^N \\ & Y \arrow[ul, "f"] \arrow[u, "G"'] \end{tikzcd}$$
exists with f birational, projective

3. a) A *log resolution* of X with a divisor $D = \sum D_i$

is a $f: Y \to X$ birational proper with

- Y smooth
- $\tilde{D} \cup \mathrm{Exc}(f)$ a snc divisor

(In particular, $\mathrm{Exc}(f)$ of codimension 1)

(Sometimes only "$\tilde{D}$ snc" is stated / proved)

b) A *strong log resolution* is a log resolution

s.t. • $f: f^{-1}(X \setminus (X_{sing} \cup D_{sing})) \xrightarrow{\sim} X \setminus (X_{sing} \cup D_{sing})$

• f is a sequence of blow-ups with smooth centers.

4 Suppose $\varphi: X \dashrightarrow X'$ is a birational map.
X, X' smooth, projective

a) A *weak factorization* of φ consists of a sequence of birational maps

$$X \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \longrightarrow X_n \overset{\varphi_n}{\dashrightarrow} X$$

with either φ_i or φ_i^{-1} a blow-up of a smooth irreducible subvariety

If $\varphi|_U$ isomorphism, then wlog $\varphi_i|_U$ iso.

b) A *strong factorization* is of the form

$$\begin{tikzcd} & & X_k \arrow[dl] \arrow[dr, "\varphi_k"] & & \\ & X_1 \arrow[dl, "\varphi_0^{-1}"'] & & \cdots \arrow[dr, dashed] & \\ X & & & & X \end{tikzcd}$$

The existence is not proved.