Singularities in dimension two

Proposition: Let X be a normal surface.

X is terminal $\iff$ X is smooth.

Proof. $\Leftarrow$ ✓

"$\Rightarrow$" We shall treat a simplified case first.

Let $f : Y \to X$ be a resolution with $\mathrm{Exc}(f) = E$ an irreducible smooth curve.

Adjunction formula: $K_E = (K_Y + E)|_E$ and hence

$$-2 \leq 2g(E) - 2 = (K_Y + E \cdot E)$$

X terminal $\Rightarrow$ $K_Y = f^*K_X + aE$ with $a > 0$

Then $(K_Y + E \cdot E) = (f^*K_X + (a+1)E \cdot E) = (a+1)E^2$

Hence $-2 \leq (a+1)E^2$

Use $E^2 < 0$ (*) to conclude $a = 1$, $E^2 = -1$, $g(E) = 0$.

Then use

Castelnuovo: If E is a (-1) curve, i.e. $E^2 = -1$, $E \cong \mathbb{P}^1$, then $f : Y \to X$ is the blow-up of a smooth point $x \in X$ (**)

(More precisely one would have

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & & \\ X_0 & & \end{array}$$

π = blow-up of X_0 smooth and $f(\pi^{-1}(x)) = pt$ $\forall x \in X_0$

Using $\pi_* \mathcal{O}_Y = \mathcal{O}_{X_0}$ and $f_* \mathcal{O}_Y = \mathcal{O}_X$, this yields factorization $X_0 \dashrightarrow X$.)

◁

The general case goes as follows.

i) Let $f: Y \to X$ be birational proper with E_i exceptional.
Then $(\sum a_i E_i)^2 < 0$ if $(a_i) \neq 0$.

If X is projective, the proof can be given by using the Hodge-Index.

Choose D ample on X $\Rightarrow D^2 > 0 \Rightarrow (f^*D)^2 > 0$.
Clearly, $(\sum a_i E_i \, . \, f^*D) = 0$. Since the intersection form on $NS(Y)$ has signature $(1, -1, \ldots, -1)$ this yields $\sum a_i E_i = 0$ or $(\sum a_i E_i)^2 < 0$.

[illegible]

ii) For the arithmetic genus $g(C) = 1 - \chi(\mathcal{O}_C)$ of an irreducible curve $C \subset Y$ one still has the adjunction formula (Riemann-Roch)

$$2g(C) - 2 = (K_Y + C \, . \, C).$$

If $\nu: \tilde{C} \to C$ is the normalization, then $g(\tilde{C}) + \delta(C) = g(C)$.

Thus, if $g(C) = 0$, then $C \cong \mathbb{P}^1$ (smooth).
Thus, the assumption E smooth was superfluous.

iii) Let $f: Y \to X$ be a minimal resolution of a normal singular surface X. Here, "minimal" means that Y does not contain any (-1)-curve E with $f(E) = pt$.
Then in the ramification formula

$$K_Y = f^*K_X + \sum a_i E_i$$ the coefficients satisfy $a_i \leq 0$.

Suppose $a_1, \dots, a_k > 0$, $a_{k+1}, \dots, a_n \le 0$

Then

$$(K_Y \cdot \sum_{i=1}^{k} a_i E_i) = (f^* K_X + \sum_{i=1}^{n} a_i E_i \cdot \sum_{i=1}^{k} a_i E_i)$$

$$= \underbrace{\left(\sum_{i=1}^{k} a_i E_i\right)^2}_{<0} + \underbrace{\left(\sum_{j=k+1}^{n} a_j E_j \cdot \sum a_i E_i\right)}_{\le 0}$$

≤ 0, since $a_j a_i < 0$ $j > k$, $i \le k$

and $(E_j \cdot E_i) \ge 0$ for $i \ne j$

Hence wlog $(K_Y \cdot E_1) < 0$.

Together with $E_1^2 < 0$ and the adjunction formula, this yields $E_1^2 = -1$, $E_1 \cong \mathbb{P}^1$, a (-1) curve.

Contradiction

Clearly i) - iii) prove the assertion. $\square$

Canonical surface singularities can also be completely classified. They form the so-called ADE singularities. They are hypersurface singularities

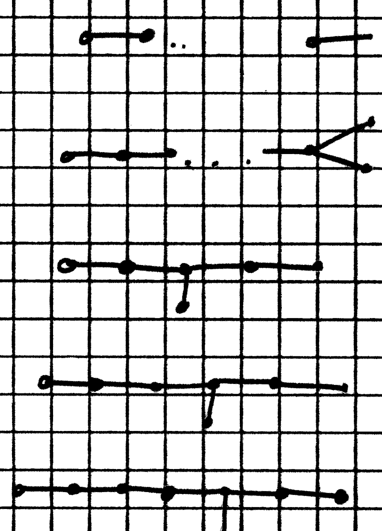
A_n $f = xy + z^{n+1}$

D_n $f = x^2 + y(z^2 + y^{n-2})$

E_6 $f = x^2 + y^3 + z^4$

E_7 $f = x^2 + y(y^2 + z^3)$

E_8 $f = x^2 + y^3 + z^5$



dual graph of minimal resolution

Canonical singularities as the singularities of the canonical model

Suppose Y smooth, projective $\rightsquigarrow \omega_Y \in \mathrm{Pic}(Y)$

Y is of general type if the canonical ring

$$R := \bigoplus_{m=0}^{\infty} H^0(Y, \omega_Y^{\otimes m})$$

has Krdeg $Q(R) - 1 = \dim Y$

(i.e. $\mathrm{Kod}(Y) = \dim(Y)$. An alternative way to define the Kodaira dimension is

$$\mathrm{Kod}(Y) = \max\{\dim(\mathrm{Im}\,\varphi_{\omega_Y^{\otimes m}})\}, \text{ where}$$

$\varphi_{\omega_Y^{\otimes m}} : Y \setminus \mathrm{Bs}|\omega_Y^{\otimes m}| \longrightarrow \mathbb{P}^N$ is the morphism induced by the linear system $\omega_Y^{\otimes m}$ $\nearrow$ [Ueno])

Suppose K_Y is nef (i.e. $(K_Y . C) \geq 0$ for any curve $C \subset Y$)

$\nearrow$ later

Then

1 Conjecture (Abundance): mK_Y is base point free for $m \gg 0$. (Also for Y with terminal singularities)

This together with other standard conjectures would yield

2. Conjecture R is always finitely generated. (Y terminal)

This is open!

1. $\Rightarrow |mK_Y|$, $m \gg 0$ yields $Y \xrightarrow{\phi} Y_{can} \subset \mathbb{P}^N$

Y_{can} is the canonical model of Y

with ϕ contracting exactly the curves C with $C . K_Y = 0$

2 $\Rightarrow$) The canonical model of Y with K_Y nef can simply be given as $\operatorname{Proj} R$, where
$R = \bigoplus H^0(Y, mK_Y)$

Roughly: Canonical singularities occur whenever one tries to pass from the minimal model (terminal + K_Y nef) to the canonical model

More precisely:

Proposition: Let X be projective . Then the following conditions are equivalent.
i) X has only canonical singularities with K_X ample
ii) $\exists$ Y smooth projective of general type s.t. $X = \operatorname{Proj} R_Y$ with R_Y the canonical ring $\bigoplus H^0(Y, \mathcal{O}(mK_Y))$ which is supposed to be finitely generated

Remark: K_X only Weil divisor, but for some $r > 0$ $\mathcal{O}(rK_X) \in \operatorname{Pic}(X)$ makes sense. One says K_X ample if $\mathcal{O}(rK_X)$ is ample
For simplicity we shall assume X Gorenstein.

Proof: "$\Longrightarrow$" Easy direction

Choose desingularization $f: Y \to X$

X canonical $\Longrightarrow K_Y = f^* K_X + \sum a_i E_i$ with $a_i \geq 0$ $(a_i \in \frac{1}{a}\mathbb{Z})$

with E_i exceptional divisors

Have earlier proved $f_* \mathcal{O}_Y(\sum b_i E_i) = \mathcal{O}_X$ whenever $b_i \geq 0$.

Hence $f_* \mathcal{O}(mK_Y) = \mathcal{O}(mK_X)$ and hence

$$R = \bigoplus H^0(Y, \mathcal{O}(mK_Y)) = \bigoplus H^0(X, f_* \mathcal{O}(mK_Y))$$

$$= \underbrace{\bigoplus H^0(X, \mathcal{O}(mK_X))}_{\text{finitely generated, because } K_X \text{ ample}}$$

and $\mathrm{Proj}(\qquad\qquad) = X$

"$\Longleftarrow$" Suppose Y of general type, $R = \bigoplus \underbrace{H^0(Y, \mathcal{O}(mK_Y))}_{=: R_m}$ finitely generated

Then there exists $d > 0$ s.t. $R_d^{\otimes m} \twoheadrightarrow R_{md}$ for all $m > 0$

$\rightsquigarrow$ $R' := \bigoplus R_{md} = \bigoplus \underbrace{H^0(Y, \mathcal{O}(mdK_Y))}_{R'_m}$.

Then R' is generated by R'_1, i.e. $R_1'^{\otimes m} \twoheadrightarrow R'_m$ and

$X = \mathrm{Proj}(R) = \mathrm{Proj}(R')$

$=$

Now use linear systems:

As $R_1'^{\otimes m} \twoheadrightarrow R'_m$ one has $Bs(R'_1) = Bs(R'_m)$ (as sets)

Let $F :=$ divisorial part of $Bs(R'_1)$ (with multiplicities)

and $M := dK_Y - F$. Then

$$\begin{array}{ccc} H^0(Y, \mathcal{O}(mM)) & \xrightarrow{\cong} & R'_m \\ D & \longmapsto & D + mF \end{array}$$

. One writes $R'_m = |mM| + mF$

Hence $X \cong \mathrm{Proj}(R') \cong \mathrm{Proj}\, \bigoplus H^0(Y, \mathcal{O}(km))$

i) **Claim**: We may choose Y such that M is base point free (i.e. $Bs(R_M')$ has pure codimension one)

General technique: Elimination of indeterminacies

$\exists$: $\pi : \tilde{Y} \to Y$ projective, birational, $\tilde{Y}$ smooth (series of smooth blow-ups) with exceptional divisors $E_1', \dots, E_k'$ s.t. $\pi^* M - \sum a_i E_i' =: \tilde{M}$ is base point free for some $a_i \geq 0$.

On the other hand: $K_{\tilde{Y}} = \pi^* K_Y + \sum b_i E_i'$ with $b_i > 0$

Thus, $d K_{\tilde{Y}} = \pi^*(M + F) + \sum b_i E_i'$
$= \tilde{M} + \tilde{F} + \sum c_i E_i'$ with $c_i > 0$

$\tilde{F}$ strict transform

Now $Bs(\tilde{M}) = \emptyset$ and $Bs(|dK_{\tilde{Y}}|) = \tilde{F} + \sum c_i E_i'$ purely of codimension one

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From now on: M base point free

$\rightsquigarrow \varphi_M : Y \to X \subset \mathbb{P}^N$ $\quad \varphi_M^* \mathcal{O}(1) = \mathcal{O}(M)$

ii) **Claim**: X is normal

If not then
$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, dotted, "\varphi'"'] & X \\ \tilde{X} \arrow[ur, "\nu"'] & \end{tikzcd}$$
ν = normalization

η finite $\Rightarrow$ $\eta^*\mathcal{O}(1)$ ample

η isom. $\Leftarrow$ $H^0(\tilde{X}, \eta^*\mathcal{O}(m)) \cong H^0(X, \mathcal{O}(m))$ $\forall m$

Now use

$H^0(Y, \mathcal{O}(mM)) \cong H^0(X, \mathcal{O}(m))$

$\hookrightarrow H^0(\tilde{X}, \eta^*\mathcal{O}(m))$

$\hookrightarrow H^0(Y, \varphi'^*\eta^*\mathcal{O}(m)) = H^0(Y, \mathcal{O}(mM))$

Hence "$\cong$" everywhere $\square$

(iii) Claim: $\underbrace{\dim(X) = \dim(Y)}$, $\operatorname{codim} \varphi(F) \geq 2$

Since Y is of general type

Suppose $\operatorname{codim} \varphi(F) = 1$. Let $G \subset F$ be an irred. component with $\varphi(G) \subset X$ divisor

$0 \to \mathcal{O} \to \mathcal{O}(G) \to \mathcal{O}(G)|_G \to 0$ yields

$0 \to \mathcal{O}(mM) \to \mathcal{O}(mM+G) \to \mathcal{O}(mM+G)|_G \to 0$ and

$$\underbrace{\overset{\displaystyle H^0(Y, \mathcal{O}(mM+G))}{\underset{\displaystyle H^0(Y, \mathcal{O}(mM+mF))}{\cap}}}_{\text{all vanish along } F} \to H^0(G, \mathcal{O}(mM+G)|_G) \to H^1(Y, \mathcal{O}(mM))$$

$\Rightarrow$ is injective

$\to H^0(G, \mathcal{O}(mM+G)|_G) \hookrightarrow H^1(Y, \mathcal{O}(mM))$

Idea: LHS grows faster than RHS $\Rightarrow$ $\lightning$.

Check: $H^1(Y, \mathcal{O}(mM))$

Use spectral sequence:

$$E_2^{pq} = H^p(X, R^q\varphi_* \mathcal{O}(mM)) \Rightarrow H^{p+q}(Y, \mathcal{O}(mM))$$

To $H^1(Y, \mathcal{O}(mM))$ only the two terms

$\underbrace{H^0(X, R^1\varphi_* \mathcal{O}(mM))}$ and $\underbrace{H^1(X, \varphi_* \mathcal{O}(mM))}$ contribute

$= H^0(X, R^1\varphi_* \mathcal{O} \otimes \mathcal{O}(mM))$ $\qquad$ $= H^1(X, \varphi_* \mathcal{O} \otimes \mathcal{O}(mM))$

$= 0$ for $m \gg 0$ (Serre)

$R^1\varphi_*$ has its support over the image of fibers of dimension at least one (at most!)

$\Rightarrow$ codim supp $R^1\varphi_* \mathcal{O} \geq 2$

$\Rightarrow \dim H^0(X, R^1\varphi_* \mathcal{O}) \otimes \mathcal{O}(m)) \leq m^{\dim(X)-2}$ + lower order

Check: $H^0(G, \mathcal{O}(mM + G)|_G)$

General: $\varphi : G \to \varphi(G) \subset \mathbb{P}^N$ generically finite

$\mathcal{L} \in \operatorname{Pic}(G) \Rightarrow \varphi_* \mathcal{L} \neq 0 \Rightarrow H^0(\varphi(G), \mathcal{O}(m_0) \otimes \varphi_* \mathcal{L}) \neq 0$

for some $m_0 \gg 0$

$\rightsquigarrow H^0(\varphi(G), \mathcal{O}(m)) \hookrightarrow H^0(G, \varphi^* \mathcal{O}(m)) \overset{\cdot s}{\hookrightarrow} \underbrace{H^0(G, \varphi^*(\mathcal{O}(m + m_0)) \otimes \mathcal{L})}$

where $0 \neq s \in H^0(G, \varphi^* \mathcal{O}(m_0) \otimes \mathcal{L})$

Hence for $m \to \infty$ the dimension of ( ) grows like $m^{\dim(G)} = m^{\dim(X)-1}$.

(In our case $\mathcal{L} = \mathcal{O}(G)|_G$.)

iv) <u>Claim</u>: φ is birational

$$\exists\, U \subseteq X \text{ open},\ \operatorname{codim}(X\setminus U) \geq 2,\ \varphi : \varphi^{-1}(U) \cong U.$$

The arguments in ii) with $\tilde{X}$ replaced by the Stein factorization show $\varphi_* \mathcal{O}_Y = \mathcal{O}_X$. Hence φ is birational. Then set

$U := X \setminus \varphi(\operatorname{Exc}(\varphi))$.

Then $\varphi^*(\mathcal{O}(1)|_U) = \mathcal{O}(m)|_{\varphi^{-1}(U)} = \mathcal{O}(dK_Y)|_{\varphi^{-1}(U)}$

$$= \mathcal{O}(dK_U)$$

Thus, for the line bundle $\mathcal{O}(1)|_X$ one has

$$\mathcal{O}(1)|_U = \mathcal{O}(dK_U) \quad \text{and} \quad \operatorname{codim} X\setminus U \geq 2$$

$$\Rightarrow \quad \mathcal{O}(1) = \mathcal{O}(dK_X)$$

In particular, X is <u>$\mathbb{Q}$-Gorenstein</u>

Moreover, $\varphi^*\mathcal{O}(dK_X) = \varphi^*\mathcal{O}(1) = \mathcal{O}(m)$

$$= \mathcal{O}(dK_Y - F),$$

i.e. $dK_Y = \varphi^*(dK_X) + \underbrace{F}_{\text{effective}}$

$\Rightarrow$ X is canonical. $\square$

(Remark: The argument does not show that X is terminal, as φ might contract other divisors. We only have $F \subset \operatorname{Exc}(\varphi)$.)