Other characterization of canonical / terminal singularities

X has only <u>canonical singularities</u>
iff
- X $\mathbb{Q}$-Gorenstein
- $\exists\, f: Y \to X$ proper, birational, Y smooth with $f_* \mathcal{O}(m K_Y) = \mathcal{O}(m K_X)$ (⊛)

(Either ⊛ for all $m = n \cdot r$, $n \in \mathbb{Z}$, where r is such that rK_X is Cartier or one works with the reflexive sheaves $\omega_X^{[m]}$)

<u>Proof</u> "⇒" $\mathbb{Q}$-Gorenstein by definition of 'canonical'.

Write $K_Y = f^* K_X + \sum a_i E_i$, E_i exceptional, $a_i \geq 0$
Hence $f_* \mathcal{O}(m K_Y) = \mathcal{O}(m K_X) \otimes f_* \mathcal{O}(\sum a_i E_i)$ by projection formula.
The natural inclusion $\mathcal{O}_Y \hookrightarrow \mathcal{O}(\sum a_i E_i)$ for $a_i \geq 0$!
yields inclusion
$$f_* \mathcal{O}_Y \longrightarrow f_* \mathcal{O}(\sum a_i E_i)$$

By Zariski Main Theorem $\mathcal{O}_X = f_* \mathcal{O}_Y$.

Moreover, $s \in \Gamma(U, f_* \mathcal{O}(\sum a_i E_i)) \rightsquigarrow s|_{U \setminus f(Exc(f))} =: \bar{s}$
is a section of $f_* \mathcal{O}(\sum a_i E_i)|_{f^{-1}(U) \setminus Exc(f)} \cong \mathcal{O}_{f^{-1}(U)}$,

i.e. $\bar{s} \in H^0(U \setminus f(Exc(f)), \mathcal{O}_X)$, which naturally extends to $\tilde{s} \in H^0(U, \mathcal{O}_X)$, for codim $f(Exc(f)) \geq 2$ and X is normal. Now compare image
$$\tilde{s} \longmapsto \tilde{s}' \qquad \text{with } s$$
$$\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U, f_* \mathcal{O}(\sum a_i E_i))$$
As $s = s'$ on $U \setminus f(Exc(f))$ resp. on $f^{-1}(U) \setminus Exc(f)$ as

sections of the line bundle, $\mathcal{O}(\sum a_i E_i)$ they coincide. i.e. $\mathcal{O}_X \to f_* \mathcal{O}(\sum a_i E_i)$ is an isomorphism. This proves $f_* \mathcal{O}(mK_Y) = \mathcal{O}(mK_X)$.

"$\Leftarrow$" Suppose $K_Y = f^* K_X + \sum_{i=1}^n a_i E_i$ with $a_1 < 0$

Then $f_* \mathcal{O}(mK_Y) \otimes \mathcal{O}(-mK_X) \cong f_* \mathcal{O}(m a_1 E_1 + \sum_{i=2}^n m a_i E_i)$

Know $f_* \mathcal{O}(\sum_{i=2}^n m a_i E_i) \cong \mathcal{O}_X$ and $1 \in \Gamma(X, \mathcal{O}_X)$

yields the section $s \in H^0(Y, \mathcal{O}(\sum_{i=2}^n m a_i E_i))$

with zero set $\sum_{i=2}^n m a_i E_i$

Consider

$$0 \to \mathcal{O}(m a_1 E_1 + \sum_{i=2}^n m a_i E_i) \longrightarrow \mathcal{O}(\sum_{i=2}^n m a_i E_i) \to \mathcal{O}(\sum_2^n \quad)|_{m a_1 E_1} \to 0$$

$$s \longmapsto \neq 0, \text{ as}$$

$E_1, \dots E_n$ distinct prime divisors.

Hence s could never be a section of $f_*(\sum_{i=1}^n m a_i E_i)$,

i.e. $f_* \mathcal{O}(\sum_{i=1}^n m a_i E_i) \not\cong \mathcal{O}_X$.

Similarly one proves:

X has only <u>terminal singularities</u>

iff:
- X $\mathbb{Q}$-Gorenstein
- $f_* \mathcal{O}(mK_Y - \sum E_i) \cong \mathcal{O}(mK_X)$

There is a way to think of K_X as a sheaf (not invertible though) and not only as a Weil divisor.
We shall provide a few hints needed to establish the following bijection:

$$\boxed{Cl(X) \cong \{\text{reflexive sheaves of rank one}\}/\cong} \quad (*)$$

for X normal and integral.

<u>Recollections</u>:

$\mathcal{F} \in Coh(X) \rightsquigarrow \mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$

$$\mathcal{F}^{\vee\vee} = (\mathcal{F}^\vee)^\vee$$

Clearly, $\mathcal{F}^\vee$ is torsion free and $\exists\ \mathcal{F} \to \mathcal{F}^{\vee\vee}$

Then $\mathcal{F}$ is called <u>reflexive</u> if $\mathcal{F} \xrightarrow{\cong} \mathcal{F}^{\vee\vee}$

For the following facts see Hartshorne Math. Ann. 254

<u>Lemma 1</u>: X normal, integral. Then the following conditions are equivalent

i) $\mathcal{F}$ reflexive

ii) $\mathcal{F}$ torsion free and normal

(i.e. $\mathcal{F}(U) \cong \mathcal{F}(U \setminus Y)$ $\forall U \subset X$ open, $\forall Y \subset U$ closed, $\operatorname{codim} Y \geq 2$)

iii) $\mathcal{F}|_U \cong i_*(\mathcal{F}|_{U \setminus Y})$, where U, Y as before

<u>Lemma 2</u>: X integral, factorial. Then any reflexive sheaf of rank one is invertible

<u>Pf</u>. Use $\mathcal{F}$ reflexive $\Rightarrow \exists\ Y \subset X_{reg}$, $\operatorname{codim} Y \geq 3$,

$\mathcal{F}|_{X_{reg} \setminus Y}$ locally free

Then $\exists\ Y \subset X$ closed, $\operatorname{codim} Y \geq 2$: $\mathcal{F}|_{X \setminus Y} \in Pic(X \setminus Y)$

Then use $Cl(X\setminus Y) \xrightarrow{\sim} Cl(X)$

$$\begin{array}{ccc} Cl(X\setminus Y) & \xrightarrow{\sim} & Cl(X) \\ \wr\| & & \wr\| \\ Pic(X\setminus Y) & & Pic(X) \end{array}$$

$\Rightarrow \exists\, \mathcal{L} \in Pic(X)$ with $\mathcal{L}_{|X\setminus Y} \cong \mathcal{F}_{|X\setminus Y}$.

$\Rightarrow \mathcal{L} = i_*(\mathcal{L}_{|X\setminus Y}) = i_*(\mathcal{F}_{|X\setminus Y}) = \mathcal{F}$. $\square$

<u>Lemma 3</u> X normal, integral. Then the following conditions are equivalent

i) $\mathcal{F}$ reflexive, rk $= 1$

ii) $\forall\, Y \subset X$ closed, codim $Y \geq 2$, $X\setminus Y$ smooth

$\Rightarrow$ · $\mathcal{F}_{|X\setminus Y}$ invertible

· $\mathcal{F} = i_*(\mathcal{F}_{|X\setminus Y})$

(The main technical fact from commutative algebra is: A local normal ring, M finitely generated module then M reflexive $\Leftrightarrow$ M torsion free and depth $M \geq 2$ if dim $A \geq 2$)

Now we explain the map $(*)$

$D = \sum n_Y Y \longmapsto \mathcal{O}(D)$ with

$\mathcal{O}(D)(U) = \{ f \in K(X) \mid v_Y(f) + n_Y \geq 0 \ \forall\, Y \cap U \neq \emptyset \}$

Then : · $\mathcal{O}(D)$ torsion free ($\subset K(X)$!)

· $\mathcal{O}(D)(U) = \mathcal{O}(D)(U\setminus Z)$ if codim $Z \geq 2$, for $Y \cap U \neq \emptyset \Leftrightarrow Y \cap (U\setminus Z) \neq \emptyset$.

For the converse use that any torsion free sheaf of rank one can be realized as a subsheaf of $K(X)$.

Application to the canonical divisor:

$K_X \rightsquigarrow \mathcal{O}(K_X)$ reflexive of rank one

$$\omega_X := \mathcal{O}(K_X)$$
$$\omega_X^{[n]} := \mathcal{O}(nK_X) \qquad (\neq \omega_X^{n})$$

<u>Facts</u>:

- X Gorenstein $\Leftrightarrow$ ω_X locally free
- X $\mathbb{Q}$-Gorenstein $\Leftrightarrow$ $\omega_X^{[n]}$ locally free for some $n>0$
- $\omega_X \cong i_*(\omega_{X_{reg}})$
- $\omega_X \cong \mathcal{H}^{-d}(\omega_X^{\bullet})$, where $d=\dim(X)$, $\omega_X^{\bullet}$ dualizing complex (Reference?)
- X is CM $\Leftrightarrow$ $\omega_X^{\bullet} \simeq \omega_X[d]$ (see Hartshorne: Residues and duality)

Proposition

X normal. Then X is canonical

$\Leftrightarrow$ • X is $\mathbb{Q}$-Gorenstein, i.e. $\omega_X^{[n]}$ locally free for some $n>0$

• If $f: Y \to X$ is a desingularization, then $\exists\ f^*\omega_X^{[n]} \to \omega_Y^{n}$ which is $=id$ on $Y \setminus Exc(f)$ for all n.

<u>Proof</u>. "$\Rightarrow$" Fix K_Y such that
$$K_Y = f^*K_X + \sum a_i E_i \quad \text{in } Z(X)$$
and not only up to linear equivalence. If $K_Y = \sum m_i Y_i$, $f^*K_X = \sum n_i Y_i$ then $m_i \geq n_i$.

Hence $\{h \in K(Y) \mid v_{Y_i}(h) + n_i \geq 0 \quad Y_i \cap U \neq \emptyset\}$
$\subset \{h \in K(Y) \mid v_{Y_i}(h) + m_i \geq 0 \quad Y_i \cap U \neq \emptyset\}$.

Thus $\mathcal{O}(\varphi^* K_X) \subset \mathcal{O}(K_Y)$ as subsheaves of $K(Y)$

Similarly for $\mathcal{O}(\varphi^* r K_X) \subset \mathcal{O}(r K_Y)$

"$\Leftarrow$" analogous $\square$